The classification of the trivectors of a 6-dimensional symplectic space: summary, consequences and connections

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Abstract

The large project of classifying all $Sp(V,f)$-equivalence classes of trivectors of a 6-dimensional symplectic space (V,f) over an arbitrary field has recently been finished by the authors. In the present paper, we give a summary of this classification, derive several consequences of it and indicate how this classification relates to the other known results from the literature.

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1 Introduction

Throughout this paper, V denotes a 6-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f. With the pair (V,f), there is associated a symplectic group $Sp(V,f) \cong Sp_6(\mathbb{F})$ which consists of all $\theta \in GL(V)$ for which $f(\bar{x}^\theta, \bar{y}^\theta) = f(\bar{x}, \bar{y})$, $\forall \bar{x}, \bar{y} \in V$.

An ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of V is called a *hyperbolic basis* of (V,f) if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i,j \in \{1,2,3\}$. Throughout this paper, $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ will denote a fixed hyperbolic basis of (V,f). The group $Sp(V,f)$ consists of those elements of $GL(V)$ that map hyperbolic bases of (V,f) to hyperbolic bases of (V,f).

The group $GL(V)$ and its subgroup $Sp(V,f)$ have a natural action on the third exterior power $\bigwedge^3 V$ of V. Indeed, for every $\theta \in GL(V)$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$ such that $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$, $\forall \bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. The elements of $\bigwedge^3 V$ are called the *trivectors* of V. Two trivectors χ_1 and χ_2 of V are called $Sp(V,f)$*-equivalent* if there exists a $\theta \in Sp(V,f)$ such that $\bigwedge^3(\theta)(\chi_1) = \chi_2$.

In a series of four papers [2, 3, 4, 5], the authors finished their large project of obtaining a classification of all $Sp(V,f)$-equivalence classes of trivectors of V without imposing any restrictions on the underlying field. The classification of the $Sp(V, f)$-equivalence classes as we will give it in Section 2 can easily be extracted from these papers. In Sections 3 and 4, we will give several consequences of this classification.

Under the assumption that the underlying field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2, a classification of the $Sp(V, f)$-equivalence classes of trivectors of V was already obtained by Popov [8, Section 3]. This classification will be recalled in Section 5 (with a few corrections) and in Section 6 we will establish explicit $Sp(V, f)$-equivalences between the trivectors in our classification and those from [8]. The determination of explicit $Sp(V, f)$-equivalences between the trivectors of both classifications was time-consuming and quite nontrivial in several cases. For this reason, we believed it would be valuable to explicitly register an identification between the trivectors of both classifications in the literature.

2 The classification of the $Sp(V, f)$-equivalence classes of trivectors

Let $\mathbb{F}$ be a field and let $\overline{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}$. We put $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$. For every separable quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, we choose $a_{\mathbb{F}'} \in \mathbb{F}$ and $b_{\mathbb{F}'} \in \mathbb{F}$ in such a way that $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $X^2 - a_{\mathbb{F}'}X - b_{\mathbb{F}'}$. In general, there are many possibilities for $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$. In the discussion below, $(a_{\mathbb{F}'}, b_{\mathbb{F}'})$ will be a fixed choice among all these possibilities. For every nonseparable quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, let $a_{\mathbb{F}'}$ be a nonsquare in $\mathbb{F}$ such that $\mathbb{F}' = \mathbb{F}[\sqrt{a_{\mathbb{F}'}}]$. There are many choices for $a_{\mathbb{F}'}$, but again $a_{\mathbb{F}'}$ will be a fixed choice among all these possibilities. Put

$$\begin{aligned} \Psi &:= \{(a_{\mathbb{F}'}, b_{\mathbb{F}'}) \,|\, \mathbb{F}' \subseteq \overline{\mathbb{F}} \text{ is a separable quadratic extension of } \mathbb{F}\}, \\ \Psi' &:= \{a_{\mathbb{F}'} \,|\, \mathbb{F}' \subseteq \overline{\mathbb{F}} \text{ is a nonseparable quadratic extension of } \mathbb{F}\}. \end{aligned}$$

The next two theorems give a complete classification of the $Sp(V, f)$-equivalence classes of trivectors of V.

Theorem 2.1 ([2, 3, 4, 5]) *Every nonzero trivector of V is $Sp(V, f)$-equivalent with (at least) one of the following trivectors:*

(A1) $\chi_{A1} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;

(A2) $\chi_{A2} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$;

(B1) $\chi_{B1} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$;

(B2) $\chi_{B2} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$;

(B3) $\chi_{B3} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(B4) $\chi_{B4}(\lambda) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$;

(B5) $\chi_{B5}(\lambda) := \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C1) $\chi_{C1}(\lambda) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C2) $\chi_{C2}(\lambda) := \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C3) $\chi_{C3}(\lambda) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C4) $\chi_{C4}(\lambda) := \bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C5) $\chi_{C5}(\lambda) := \bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ *for some* $\lambda \in \mathbb{F}^*$*;*

(C6) $\chi_{C6}(\lambda, \epsilon) := \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$ *and some* $\epsilon \in \mathbb{F} \setminus \{0, -1\}$*;*

(D1) $\chi_{D1} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$*;*

(D2) $\chi_{D2}(\lambda) := \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(D3) $\chi_{D3}(\lambda_1, \lambda_2) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$*;*

(D4) $\chi_{D4}(\lambda_1, \lambda_2) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$*;*

(D5) $\chi_{D5}(\lambda) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$*;*

(D6) *(only if* $\mathsf{char}(\mathbb{F}) \neq 2$*)* $\chi_{D6} := -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$*;*

(D7) *(only if* $|\mathbb{F}| = 2$*)* $\chi_{D7} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$*;*

(E1) $\chi_{E1}(a, b, h_1, h_2, h_3) := 2 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + a \cdot \Big(h_1 \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + h_2 \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + h_3 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*\Big) + (a^2 + 2b) \cdot \Big(h_1 h_2 \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + h_1 h_3 \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + h_2 h_3 \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*\Big) + h_1 h_2 h_3 a(a^2 + 3b) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $(a, b) \in \Psi$ *and some* $h_1, h_2, h_3 \in \mathbb{F}^*$*;*

(E2) $\chi_{E2}(a, b, k) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big)$ *for some* $(a, b) \in \Psi$ *and some* $k \in \mathbb{F}^*$*;*

(E3) $\chi_{E3}(a, b, k, h) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big) + h \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $(a, b) \in \Psi$ *and some* $k, h \in \mathbb{F}^*$*;*

(E4) $\chi_{E4}(a, b, k, h_1, h_2) := (1 - h_1 h_2(a^2 + 4b)) \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + (1 + h_1 h_2(a^2 + 4b)) \cdot \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b(1 - h_1 h_2(a^2 + 4b)) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big) + h_1(1 - h_1 h_2(a^2 + 4b)) \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + (a^2 + 4b) h_2 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$ *for some* $(a, b) \in \Psi$ *and some* $k, h_1, h_2 \in \mathbb{F}^*$ *satisfying* $h_1 h_2(a^2 + 4b) \neq 1$*;*

(E5) $\chi_{E5}(a,b,k) := \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + 2 \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_2^* - a \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + a \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + (a^2+b) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + k \cdot \Big(a \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* - \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big)$ *for some* $(a,b) \in \Psi$ *and some* $k \in \mathbb{F}^*$*;*

(E1′) $\chi'_{E1}(a,h_1,h_2,h_3) := \frac{a+1}{a} \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + (a+1)h_1h_2h_3 \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + (\bar{e}_1^* + h_1\bar{f}_1^*) \wedge (\bar{e}_2^* + h_2\bar{f}_2^*) \wedge (\bar{e}_3^* + h_3\bar{f}_3^*)$ *for some* $a \in \Psi'$ *and some* $h_1,h_2,h_3 \in \mathbb{F}^*$*;*

(E2′) $\chi'_{E2}(a,k,h_1,h_2) := \frac{1}{a} \cdot \bar{e}_1^* \wedge (\bar{e}_2^* + h_1(a+1)\bar{f}_3^*) \wedge \bar{f}_2^* + k \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge (h_2(a+1)\bar{e}_2^* + \bar{f}_3^*) + \frac{1}{(a+1)^2} \cdot (\bar{e}_1^* + k\bar{f}_1^*) \wedge (\bar{e}_2^* + (a+1)\bar{e}_3^* + h_1(a+1)\bar{f}_3^*) \wedge (h_2(a+1)\bar{e}_2^* + (a+1)\bar{f}_2^* + \bar{f}_3^*)$ *for some* $a \in \Psi'$ *and some* $k,h_1,h_2 \in \mathbb{F}$ *satisfying* $k \neq 0$ *and* $h_1h_2(a+1)^2 \neq 1$*;*

(E3′) $\chi'_{E3}(a,h_1,h_2) := \frac{1}{a} \cdot \bar{e}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge (\bar{e}_1^* + h_1\bar{f}_3^*) + \frac{1}{a+1} \cdot (\bar{e}_1^* + \bar{e}_2^*) \wedge (\bar{e}_3^* + h_1\bar{f}_3^*) \wedge \Big((a+1)^2h_2\bar{e}_1^* + \bar{f}_1^* + \bar{f}_2^*\Big)$ *for some* $a \in \Psi'$*, some* $h_1 \in \mathbb{F}^*$ *and some* $h_2 \in \mathbb{F}$*.*

A trivector of V is said to be of *Type* $(X) \in \{(A1),(A2),\ldots,(E2'),(E3')\}$ if it is $Sp(V,f)$-equivalent with (one of) the trivector(s) mentioned in (X) of Theorem 2.1.

Theorem 2.1 divides the nonzero trivectors of V into 28 families. The first item of the next theorem says that trivectors belonging to distinct families can never be $Sp(V,f)$-equivalent. In the theorem, we also give necessary and sufficient conditions for two trivectors of the same family to be $Sp(V,f)$-equivalent.

Theorem 2.2 ([2, 3, 4, 5])

- *Let* $(X),(Y) \in \{(A1),(A2),\ldots,(E3')\}$ *with* $(X) \neq (Y)$*. Then no trivector of Type (X) is* $Sp(V,f)$*-equivalent with a trivector of Type (Y).*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{B4}(\lambda)$ *and* $\chi_{B4}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\frac{\lambda}{\lambda'}$ *is a square in* $\mathbb{F}$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{B5}(\lambda)$ *and* $\chi_{B5}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda = \lambda'$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{C1}(\lambda)$ *and* $\chi_{C1}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda' \in \{\lambda,-\lambda\}$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{C2}(\lambda)$ *and* $\chi_{C2}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda = \lambda'$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{C3}(\lambda)$ *and* $\chi_{C3}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda' \in \{\lambda,-\lambda\}$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{C4}(\lambda)$ *and* $\chi_{C4}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda' \in \{\lambda,-\lambda\}$*.*
- *Let* $\lambda,\lambda' \in \mathbb{F}^*$*. Then the two trivectors* $\chi_{C5}(\lambda)$ *and* $\chi_{C5}(\lambda')$ *of* V *are* $Sp(V,f)$*-equivalent if and only if* $\lambda' \in \{\lambda,-\lambda\}$*.*

- *Let $\lambda, \lambda' \in \mathbb{F}^*$ and $\epsilon, \epsilon' \in \mathbb{F}\setminus\{0,-1\}$. Then the two trivectors $\chi_{C6}(\lambda,\epsilon)$ and $\chi_{C6}(\lambda',\epsilon')$ of V are $Sp(V,f)$-equivalent if and only if $\epsilon = \epsilon'$ and $\lambda' \in \{\lambda, -\lambda\}$.*

- *Let $\lambda, \lambda' \in \mathbb{F}^*$. Then the two trivectors $\chi_{D2}(\lambda)$ and $\chi_{D2}(\lambda')$ of V are $Sp(V,f)$-equivalent if and only if $\lambda = \lambda'$.*

- *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. Then the two trivectors $\chi_{D3}(\lambda_1,\lambda_2)$ and $\chi_{D3}(\lambda_1',\lambda_2')$ of V are $Sp(V,f)$-equivalent if and only if the matrices $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_1\lambda_2)$ and $\mathrm{diag}(\lambda_1',\lambda_2',\lambda_1'\lambda_2')$ are congruent, i.e. if and only if there exists a nonsingular (3×3)-matrix A over $\mathbb{F}$ such that $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_1\lambda_2) = A \cdot \mathrm{diag}(\lambda_1',\lambda_2',\lambda_1'\lambda_2') \cdot A^T$.*

- *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}^*$. Then the two trivectors $\chi_{D4}(\lambda_1,\lambda_2)$ and $\chi_{D4}(\lambda_1',\lambda_2')$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_1'$ and there exist $X, Y \in \mathbb{F}$ such that $X^2 + \lambda_1 XY + \lambda_1 Y^2 = \frac{\lambda_2'}{\lambda_2}$.*

- *Let $\lambda, \lambda' \in \mathbb{F}^*$. If $\mathsf{char}(\mathbb{F}) \neq 2$, then the two trivectors $\chi_{D5}(\lambda)$ and $\chi_{D5}(\lambda')$ of V are always $Sp(V,f)$-equivalent. If $\mathsf{char}(\mathbb{F}) = 2$, then the two trivectors $\chi_{D5}(\lambda)$ and $\chi_{D5}(\lambda')$ of V are $Sp(V,f)$-equivalent if and only if $\frac{\lambda+\lambda'}{\lambda\lambda'}$ is of the form X^2+X for some $X \in \mathbb{F}$.*

- *Let $h_1, h_2, h_3, h_1', h_2', h_3' \in \mathbb{F}^*$ and $(a,b), (a',b') \in \Psi$. Then the two trivectors $\chi_{E1}(a,b,h_1,h_2,h_3)$ and $\chi_{E1}(a',b',h_1',h_2',h_3')$ of V are $Sp(V,f)$-equivalent if and only if $(a,b) = (a',b')$ and there exists a 3×3-matrix A over $\mathbb{F}'$ with determinant equal to 1 such that $A\cdot \mathrm{diag}(h_1,h_2,h_3)\cdot (A^{\psi})^T$ is equal to $\mathrm{diag}(h_1',h_2',h_3')$ or $\mathrm{diag}(-h_1',-h_2',-h_3')$. Here, $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ determined by the irreducible quadratic polynomial $X^2 - aX - b$ of $\mathbb{F}[X]$ and ψ is the unique nontrivial element of the Galois group $Gal(\mathbb{F}'/\mathbb{F})$.*

- *Let $k, k' \in \mathbb{F}^*$ and $(a,b), (a',b') \in \Psi$. Then the two trivectors $\chi_{E2}(a,b,k)$ and $\chi_{E2}(a',b',k')$ of V are $Sp(V,f)$-equivalent if and only if $(a,b) = (a',b')$ and $k' \in \{k,-k\}$.*

- *Let $k, h, k', h' \in \mathbb{F}^*$ and $(a,b), (a',b') \in \Psi$. Then the two trivectors $\chi_{E3}(a,b,k,h)$ and $\chi_{E3}(a',b',k',h')$ of V are $Sp(V,f)$-equivalent if and only if $(a,b) = (a',b')$ and there exists a $\sigma \in \{1,-1\}$ and $X, Y \in \mathbb{F}$ such that $k' = \sigma k$ and $h' = \sigma h(X^2 + aXY - bY^2)$.*

- *Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}^*$ and $(a,b), (a',b') \in \Psi$ such that $h_1h_2(a^2+4b) \neq 1 \neq h_1'h_2'((a')^2+4b')$. Then the two trivectors $\chi_{E4}(a,b,k,h_1,h_2)$ and $\chi_{E4}(a',b',k',h_1',h_2')$ of V are $Sp(V,f)$-equivalent if and only if $(a,b) = (a',b')$, $h_1h_2 = h_1'h_2'$ and there exist $X, Y, Z, U \in \mathbb{F}$ and a $\sigma \in \{1,-1\}$ such that $k' = \sigma k$ and $\sigma h_1' = h_1(X^2 + aXY - bY^2) + h_2(Z^2 + aZU - bU^2)$.*

- *Let $k, k' \in \mathbb{F}^*$ and $(a,b), (a',b') \in \Psi$. Then the two trivectors $\chi_{E5}(a,b,k)$ and $\chi_{E5}(a',b',k')$ of V are $Sp(V,f)$-equivalent if and only if $(a,b) = (a',b')$ and $k' \in \{k,-k\}$.*

- *Let $h_1, h_2, h_3, h_1', h_2', h_3' \in \mathbb{F}^*$ and $a, a' \in \Psi'$. Then the two trivectors $\chi'_{E1}(a, h_1, h_2, h_3)$ and $\chi'_{E1}(a', h_1', h_2', h_3')$ of V are $Sp(V, f)$-equivalent if and only if $a = a'$ and there exists a (3×3)-matrix A over $\mathbb{F}'$ with determinant 1 such that $\mathrm{diag}(h_1', h_2', h_3') = A \cdot \mathrm{diag}(h_1, h_2, h_3) \cdot A^T$. Here, $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ determined by the irreducible quadratic polynomial $X^2 + a$ of $\mathbb{F}[X]$.*

- *Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}$ and $a, a' \in \Psi'$ such that $k \neq 0 \neq k'$ and $h_1 h_2 (a+1)^2 \neq 1 \neq h_1' h_2' (a'+1)^2$. Then the two trivectors $\chi'_{E2}(a, k, h_1, h_2)$ and $\chi'_{E2}(a', k', h_1', h_2')$ of V are $Sp(V, f)$-equivalent if and only if $a = a'$, $k = k'$, $h_1 h_2 = h_1' h_2'$ and there exist $X, Y, Z, U \in \mathbb{F}$ such that $h_1' = h_1(X^2 + aY^2) + h_2(Z^2 + aU^2) + (XU + YZ)$.*

- *Let $h_1, h_2, h_1', h_2' \in \mathbb{F}$ and $a, a' \in \Psi'$ such that $h_1 \neq 0 \neq h_1'$. Then the two trivectors $\chi'_{E3}(a, h_1, h_2)$ and $\chi'_{E3}(a', h_1', h_2')$ of V are $Sp(V, f)$-equivalent if and only if $a = a'$, $h_1 = h_1'$ and $h_2 + h_2'$ is of the form $h_1(X^2 + aY^2) + Y$ for some $X, Y \in \mathbb{F}$.*

In Theorem 2.2, $\mathrm{diag}(h_1, h_2, h_3)$ denotes the (3×3)-diagonal matrix whose (i, i)-th entry is equal to h_i for every $i \in \{1, 2, 3\}$.

Remark 2.3 The trivectors mentioned as Type (D7) in Theorem 2.1 only exist if $|\mathbb{F}| = 2$, which seems somewhat unnatural. However, in the case where $\mathsf{char}(\mathbb{F}) = 2$, a "more natural" classification can be given if we replace (D5) and (D7) by the following:

($D5'$) $\chi_{D5}(\lambda) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F}^*$ such that λ^{-1} is not of the form $X^2 + X$ for some $X \in \mathbb{F}$;

($D7'$) (only if $\mathsf{char}(\mathbb{F}) = 2$) $\chi_{D7} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$.

Indeed, in De Bruyn and Kwiatkowski [4, Lemma 5.20], it was shown that if $\mathsf{char}(\mathbb{F}) = 2$, then the trivectors $\chi_{D5}(\lambda)$ and χ_{D7} are $Sp(V, f)$-equivalent for every $\lambda \in \mathbb{F}^*$ such that λ^{-1} is of the form $X^2 + X$ for some $X \in \mathbb{F}$.

3 The classification of the $Sp(V, f)$-equivalence classes of trivectors contained in a certain 14-dimensional subspace W of $\bigwedge^3 V$

The subspaces $W := < \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \bar{e}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 - \bar{e}_1 \wedge \bar{f}_1), \bar{f}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 - \bar{e}_1 \wedge \bar{f}_1), \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2) >$ and $\widetilde{W} := < \bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \bar{e}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_1), \bar{f}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_1), \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2) >$ of $\bigwedge^3 V$ are independent of the considered hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) and hence are stabilized by the natural action of $Sp(V, f)$ on $\bigwedge^3 V$. If $\mathbb{F}$ is a field of characteristic distinct from 2, then $\bigwedge^3 V = W \oplus \widetilde{W}$. If $\mathsf{char}(\mathbb{F}) = 2$, then $\widetilde{W} \subseteq W$.

If $\mathsf{char}(\mathbb{F}) \neq 2$, then every trivector $\chi \in \bigwedge^3 V$ can be written in a unique way as $\chi_1 + \chi_2$ where $\chi_1 \in W$ and $\chi_2 \in \widetilde{W}$. We define $\pi_W(\chi) := \chi_1$ and $\pi_{\widetilde{W}}(\chi) := \chi_2$.

If $\bar{v}$ is a nonzero vector of V, then we define

$$\phi(\bar{v}) := \bar{v} \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3),$$

where the vectors $\bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3$ are chosen in such a way that $(\bar{v}, \bar{w}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is a hyperbolic basis of (V, f) for a certain vector $\bar{w} \in V$. It is possible to prove (see e.g. [1, Lemma 3.9]) that $\phi(\bar{v})$ is independent of the chosen hyperbolic basis $(\bar{v}, \bar{w}, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f). We also put $\phi(\bar{o})$ equal to the zero vector of $\bigwedge^3 V$. Then $\phi : V \to \widetilde{W}$ is a linear isomorphism between the 6-dimensional vector spaces V and $\widetilde{W}$, see e.g. [1, Lemma 3.10]. If $\theta \in Sp(V, f)$, then $\phi \circ \theta = \bigwedge^3(\theta) \circ \phi$. This implies that there is only one orbit of the natural action of $Sp(V, f)$ on the nonzero vectors of $\widetilde{W}$. The following corollary of Theorem 2.1 gives (in combination with the relevant entries of Theorem 2.2) a complete classification of the $Sp(V, f)$-equivalence classes of nonzero trivectors contained in W.

Corollary 3.1 *Every nonzero trivector of W is $Sp(V, f)$-equivalent with at least one of the following trivectors:*

- χ_{A1};
- *(only if* $\mathsf{char}(\mathbb{F}) = 2$*)* χ_{B3};
- $\chi_{B4}(\lambda)$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C1}(\lambda)$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{D3}(\lambda_1, \lambda_2)$ *for some* $\lambda_1, \lambda_2 \in \mathbb{F}^*$;
- $\chi_{E1}(a, b, h_1, h_2, h_3)$ *for some* $(a, b) \in \Psi$ *and some* $h_1, h_2, h_3 \in \mathbb{F}^*$;
- $\chi'_{E1}(a, h_1, h_2, h_3)$ *for some* $a \in \Psi'$ *and some* $h_1, h_2, h_3 \in \mathbb{F}^*$.

4 The classification of the $Sp(V, f)$-equivalence classes of trivectors for some special cases of $\mathbb{F}$

In Theorems 2.1 and 2.2 and Corollary 3.1, we gave classification results for $Sp(V, f)$-equivalence classes of trivectors. The next three corollaries simplify these classification results in the special case the underlying field has no quadratic extensions.

Corollary 4.1 *Suppose $\mathbb{F}$ is a field over which there are no irreducible quadratic polynomials. Then every nonzero trivector of V is $Sp(V, f)$-equivalent with (at least) one of the following trivectors:*

- $\chi_{A1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;
- $\chi_{A2} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$;
- $\chi_{B1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$;
- $\chi_{B2} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$;
- $\chi_{B3} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;
- $\chi_{B4}(1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$;
- $\chi_{B5}(\lambda) = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ *for some* $\lambda \in \mathbb{F}^*$;

- $\chi_{C1}(\lambda) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C2}(\lambda) = \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C3}(\lambda) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C4}(\lambda) = \bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C5}(\lambda) = \bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{C6}(\lambda, \epsilon) = \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $(\lambda, \epsilon) \in \mathbb{F}^* \times (\mathbb{F}^* \setminus \{-1\})$;
- $\chi_{D1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$;
- $\chi_{D2}(\lambda) = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{D3}(1,1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$;
- $\chi_{D4}(\lambda, 1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}^*$;
- $\chi_{D5}(1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$;
- *(only if* $\mathsf{char}(\mathbb{F}) \neq 2$*)* $\chi_{D6} = -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$.

Corollary 4.2 *Suppose $\mathbb{F}$ is a field over which there are no irreducible quadratic polynomials.*

- *Let (X) and (Y) belong to $\{(A1),(A2),(B1),(B2),(B3),(B4),(B5),(C1),(C2),(C3),(C4),(C5),(C6),(D1),(D2),(D3),(D4),(D5),(D6)\}$ with $(X) \neq (Y)$. Then no trivector of Type (X) is $Sp(V,f)$-equivalent with a trivector of Type (Y).*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{B5}(\lambda_1)$ and $\chi_{B5}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_2$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{C1}(\lambda_1)$ and $\chi_{C1}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_2 \in \{\lambda_1, -\lambda_1\}$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{C2}(\lambda_1)$ and $\chi_{C2}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_2$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{C3}(\lambda_1)$ and $\chi_{C3}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_2 \in \{\lambda_1, -\lambda_1\}$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{C4}(\lambda_1)$ and $\chi_{C4}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_2 \in \{\lambda_1, -\lambda_1\}$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{C5}(\lambda_1)$ and $\chi_{C5}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_2 \in \{\lambda_1, -\lambda_1\}$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$ and $\epsilon_1, \epsilon_2 \in \mathbb{F} \setminus \{0,-1\}$. Then the two trivectors $\chi_{C6}(\lambda_1, \epsilon_1)$ and $\chi_{C6}(\lambda_2, \epsilon_2)$ of V are $Sp(V,f)$-equivalent if and only if $\epsilon_1 = \epsilon_2$ and $\lambda_2 \in \{\lambda_1, -\lambda_1\}$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{D2}(\lambda_1)$ and $\chi_{D2}(\lambda_2)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_2$.*
- *Let $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Then the two trivectors $\chi_{D4}(\lambda_1, 1)$ and $\chi_{D4}(\lambda_2, 1)$ of V are $Sp(V,f)$-equivalent if and only if $\lambda_1 = \lambda_2$.*

Corollary 4.3 *Suppose $\mathbb{F}$ is a field over which there are no irreducible quadratic polynomials. Then every nonzero trivector of V contained in W is $Sp(V,f)$-equivalent with one of the following trivectors:*

- $\chi_{A1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;
- *(only if* $\mathsf{char}(\mathbb{F}) = 2$*)* $\chi_{B3} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;
- $\chi_{B4}(1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$;
- $\chi_{C1}(\lambda) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$;

- $\chi_{D3}(1,1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$.

The following theorem was proved by Igusa [6, Proposition 7, page 1026].

Theorem 4.4 ([6]) *Suppose $\mathbb{F}$ is an algebraically closed field of characteristic distinct from 2. Then every nonzero trivector of V contained in W is $Sp(V,f)$-equivalent with one of the following trivectors:* • $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$; • $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}^*$; • $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$; • $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$.

The equivalence of Corollary 4.3 and Theorem 4.4 in the case the field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2 follows from the following two propositions.

Proposition 4.5 *If $\mathbb{F}$ is a field of characteristic distinct from 2, then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$ are $Sp(V,f)$-equivalent.*

Proof. We have $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* = \bar{e}_3^* \wedge (\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_2^* \wedge \bar{f}_2^*) = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\bar{e}_3^*, \bar{f}_3^*, \frac{1}{2}(\bar{e}_1^* + \bar{e}_2^*), \bar{f}_1^* + \bar{f}_2^*, \bar{f}_1^* - \bar{f}_2^*, \frac{1}{2}(\bar{e}_2^* - \bar{e}_1^*))$ of (V,f). ■

Proposition 4.6 *Let $\mathbb{F}$ be a field of characteristic distinct from 2 in which -1 is a square. Then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ and $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$ are $Sp(V,f)$-equivalent.*

Proof. Let $\eta \in \mathbb{F}$ such that $\eta^2 = -1$. Let θ be the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(\bar{e}_1^* + \eta \bar{e}_2^*, \frac{1}{2}\bar{f}_1^* - \frac{\eta}{2}\bar{f}_2^*, -\frac{1}{2}\bar{e}_1^* + \frac{\eta}{2}\bar{e}_2^*, -\bar{f}_1^* - \eta \bar{f}_2^*, \eta \bar{e}_3^*, -\eta \bar{f}_3^*)$ of (V,f). Then $\bigwedge^3(\theta)$ maps the trivector $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$ to the trivector $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$. ■

In the following proposition, we will treat the case where $\mathbb{F}$ is either $\mathbb{R}$ or a finite field.

Proposition 4.7 *Let V be a 6-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f.*

(1) *If $\mathbb{F} = \mathbb{R}$, then every nonzero trivector of V is $Sp(V,f)$-equivalent with precisely one of the trivectors mentioned in Table 1. In Table 1, $k^* \in \mathbb{R}^*$, $k^+, h^+ \in \{x \in \mathbb{R} \,|\, x > 0\}$, $l_1 \in \mathbb{R} \setminus \{0,-1\}$, $l_2 \in \,]0,4]$ and $l_3 \in \{x \in \mathbb{R} \,|\, x < 0$ and $x \neq -\frac{1}{4}\}$.*

(2) *Suppose $\mathbb{F} = \mathbb{F}_q$ where q is an even prime power. Let $\eta \in \mathbb{F}_q$ such that the quadratic polynomial $X^2 + X + \eta \in \mathbb{F}_q[X]$ is irreducible. Then every nonzero trivector of V is $Sp(V,f)$-equivalent with precisely one of the trivectors mentioned in Table 2. In Table 2, $k^* \in \mathbb{F}_q^*$ and $l \in \mathbb{F}_q \setminus \{0,1\}$.*

(3) *Suppose $\mathbb{F} = \mathbb{F}_q$ where q is an odd prime power. Let ν be a given nonsquare of $\mathbb{F}_q$ and let A be a subset of size $\frac{q-1}{2}$ of $\mathbb{F}_q^*$ such that for every $x \in \mathbb{F}_q^*$, precisely one of x, $-x$ belongs to A. Then every nonzero trivector of V is $Sp(V,f)$-equivalent with precisely one of the trivectors mentioned in Table 3. In Table 3, $k^* \in \mathbb{F}_q^*$, $a \in A$, $l_1 \in \mathbb{F}_q \setminus \{0,-1\}$ and $l_2 \in \mathbb{F}_q \setminus \{0, \frac{1}{4\nu}\}$.*

χ_{A1}	$\chi_{B4}(1)$	$\chi_{C3}(k^+)$	$\chi_{D2}(k^*)$	$\chi_{D5}(1)$	$\chi_{E4}(0,-1,k^*,h^+,1)$
χ_{A2}	$\chi_{B4}(-1)$	$\chi_{C4}(k^+)$	$\chi_{D3}(1,1)$	χ_{D6}	$\chi_{E4}(0,-1,k^+,l_3,1)$
χ_{B1}	$\chi_{B5}(k^*)$	$\chi_{C5}(k^+)$	$\chi_{D3}(-1,-1)$	$\chi_{E1}(0,-1,k^*,1,1)$	$\chi_{E5}(0,-1,k^+)$
χ_{B2}	$\chi_{C1}(k^+)$	$\chi_{C6}(k^+,l_1)$	$\chi_{D4}(k^*,1)$	$\chi_{E2}(0,-1,k^+)$	–
χ_{B3}	$\chi_{C2}(k^*)$	χ_{D1}	$\chi_{D4}(l_2,-1)$	$\chi_{E3}(0,-1,k^*,1)$	–

Table 1: The trivectors of a 6-dimensional real symplectic vector space.

Proof. This follows from Theorem 2.1, Theorem 2.2, Sylvester's laws of inertia (for Hermitian and real symmetric matrices), and the following known properties of $\mathbb{R}$ and the finite fields (Lidl and Niederreiter [7]).

(a) If $\lambda \in \mathbb{R}$ such that either $\lambda < 0$ or $\lambda > 4$, then $X^2 + \lambda XY + \lambda Y^2$ with $X, Y \in \mathbb{R}$ reaches all values in $\mathbb{R}$. On the other hand, if $\lambda \in \,]0, 4]$ then $X^2 + \lambda XY + \lambda Y^2$ with $X, Y \in \mathbb{R}$ reaches only the nonnegative real numbers.

(b) If q is even, then every element of $\mathbb{F}_q$ is a square. If q is odd, then the product of two nonsquares of $\mathbb{F}_q$ is a square and hence $\mathbb{F}_q = \{x^2, \nu x^2 \,|\, x \in \mathbb{F}_q\}$.

(c) If q is even, then $\{x^2 + x \,|\, x \in \mathbb{F}_q\}$ is a subgroup of index 2 of the additive group of $\mathbb{F}_q$ and so by Theorem 2.2 every trivector of Type (D5) is $Sp(V, f)$-equivalent with either $\chi_{D5}(\eta^{-1})$ or $\chi_{D5}(\eta_1^{-1})$, where η_1 is an arbitrary element of the form $x^2 + x$, $x \in \mathbb{F}_q \setminus \{0, 1\}$. Observe that η_1 only exists if $q \neq 2$. If $q \neq 2$, then $\chi_{D5}(\eta_1^{-1})$ is $Sp(V, f)$-equivalent with χ_{D7}, see Remark 2.3.

(d) Suppose q is odd. Then there exist $\lambda_1, \lambda_2 \in \mathbb{F}_q$ such that $\nu = \lambda_1^2 + \lambda_2^2$. The matrices $\mathrm{diag}(\nu, \nu)$ and $\mathrm{diag}(1, 1)$ are congruent since

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} = \begin{bmatrix} \nu & 0 \\ 0 & \nu \end{bmatrix}.$$

(e) Every polynomial $X^2 + \alpha XY + \beta Y^2 \in \mathbb{F}_q[X, Y]$ that is not a square reaches all values of $\mathbb{F}_q$. In particular, this holds for the polynomials $X^2 - \nu Y^2$ and $X^2 + \lambda_1 XY + \lambda_1 Y^2$, $\lambda_1 \notin \{0, 4\}$, of $\mathbb{F}_q$ when q is odd. If q is odd and $\lambda_1 = 4$, then $X^2 + \lambda_1 XY + \lambda_1 Y^2$ only reaches the square values of $\mathbb{F}_q$.

(f) The unique nontrivial element ψ of the Galois group $Gal(\mathbb{F}_{q^2}/\mathbb{F}_q)$ maps each $x \in \mathbb{F}_{q^2}$ to x^q. Moreover, $X^{\psi+1}$, with $X \in \mathbb{F}_{q^2}$, reaches all values of $\mathbb{F}_q$. ■

5 Popov's classification of the trivectors

In the case the characteristic of $\mathbb{F}$ is distinct from 2, the $Sp(V, f)$-equivalence classes of trivectors of V could possibly also be classified in the following way.

For some suitable index set I, let $\{\chi_i \,|\, i \in I\} \subseteq W$ be such that every trivector of V contained in W is $Sp(V, f)$-equivalent with precisely one χ_i, $i \in I$. For every $i \in I$, let

χ_{A1}	χ_{B3}	$\chi_{C2}(k^*)$	$\chi_{C6}(k^*, l)$	$\chi_{D4}(k^*, 1)$	$\chi_{E2}(1, \eta, k^*)$
χ_{A2}	$\chi_{B4}(1)$	$\chi_{C3}(k^*)$	χ_{D1}	$\chi_{D5}(\eta^{-1})$	$\chi_{E3}(1, \eta, k^*, 1)$
χ_{B1}	$\chi_{B5}(k^*)$	$\chi_{C4}(k^*)$	$\chi_{D2}(k^*)$	χ_{D7}	$\chi_{E4}(1, \eta, k^*, l, 1)$
χ_{B2}	$\chi_{C1}(k^*)$	$\chi_{C5}(k^*)$	$\chi_{D3}(1, 1)$	$\chi_{E1}(1, \eta, k^*, 1, 1)$	$\chi_{E5}(1, \eta, k^*)$

Table 2: The trivectors of a 6-dimensional symplectic $\mathbb{F}_q$-vector space, q even.

χ_{A1}	$\chi_{B4}(1)$	$\chi_{C3}(a)$	$\chi_{D2}(k^*)$	χ_{D6}	$\chi_{E5}(0, \nu, a)$
χ_{A2}	$\chi_{B4}(\nu)$	$\chi_{C4}(a)$	$\chi_{D3}(1, 1)$	$\chi_{E1}(0, \nu, a, 1, 1)$	$-$
χ_{B1}	$\chi_{B5}(k^*)$	$\chi_{C5}(a)$	$\chi_{D4}(k^*, 1)$	$\chi_{E2}(0, \nu, a)$	$-$
χ_{B2}	$\chi_{C1}(a)$	$\chi_{C6}(a, l_1)$	$\chi_{D4}(4, \nu)$	$\chi_{E3}(0, \nu, a, 1)$	$-$
χ_{B3}	$\chi_{C2}(k^*)$	χ_{D1}	$\chi_{D5}(1)$	$\chi_{E4}(0, \nu, a, l_2, 1)$	$-$

Table 3: The trivectors of a 6-dimensional symplectic $\mathbb{F}_q$-vector space, q odd.

G_i be the subgroup of $Sp(V, f)$ consisting of all $\theta \in Sp(V, f)$ for which $\bigwedge^3(\theta)$ fixes χ_i, and label the orbits of G_i on V by means of an index set J_i. For every $i \in I$ and every $j \in J_i$, let $\bar{v}^*_{i,j}$ denote an arbitrary vector of the orbit of G_i on V corresponding to j and define $\chi^*_{i,j} := \chi_i + \phi(\bar{v}^*_{i,j})$. Then for every trivector χ of V, there exist a unique $i \in I$ and a unique $j \in J_i$ such that χ and $\chi^*_{i,j}$ are $Sp(V, f)$-equivalent.

Using the above procedure and relying on Igusa's classification (as mentioned in Theorem 4.4), Popov [8, Section 3] obtained a classification of all $Sp(V, f)$-equivalence classes of trivectors of V under the assumption that the underlying field $\mathbb{F}$ is algebraically closed of characteristic distinct from 2. In the classification in [8], there are a number of inaccuracies. The trivectors $\bar{e}^*_1 \wedge \bar{f}^*_1 \wedge \bar{e}^*_3 + \bar{f}^*_2 \wedge \bar{e}^*_2 \wedge \bar{e}^*_3 + \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_3 + \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2 + \bar{e}^*_1 \wedge \bar{e}^*_3 \wedge \bar{f}^*_3$ and $\bar{e}^*_1 \wedge \bar{f}^*_1 \wedge \bar{e}^*_3 + \bar{f}^*_2 \wedge \bar{e}^*_2 \wedge \bar{e}^*_3 + \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_3 + q \cdot \bar{e}^*_3 \wedge \bar{e}^*_1 \wedge \bar{f}^*_1 + q \cdot \bar{e}^*_3 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2$, $q \in \mathbb{F}^*$, are missing in the discussion in [8]. Also, there are more $Sp(V, f)$-equivalences between trivectors than just those stated in [8][1]. In the two theorems below, we recall the classification given in [8], but with these inaccuracies already corrected.

Theorem 5.1 ([8]) *Suppose $\mathbb{F}$ is an algebraically closed field of characteristic distinct from 2. Then every trivector of V is $Sp(V, f)$-equivalent with one of the following trivectors:*

$(P1)$ $\chi_{P1} := 0$;

$(P2)$ $\chi_{P2} := \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2 + \bar{e}^*_1 \wedge \bar{e}^*_3 \wedge \bar{f}^*_3$;

$(P3)$ $\chi_{P3}(q) := \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{e}^*_3 + q \cdot \bar{f}^*_1 \wedge \bar{f}^*_2 \wedge \bar{f}^*_3$ *for some* $q \in \mathbb{F}^*$;

[1]Besides those mentioned in [8], the following additional $Sp(V, f)$-equivalences hold for every $q \in \mathbb{F}^*$: $\chi_{P3}(q) \cong \chi_{P3}(-q)$, $\chi_{P5}(q) \cong \chi_{P5}(-q)$, $\chi_{P16}(q) \cong \chi_{P16}(-q)$, $\chi_{P17}(q) \cong \chi_{P17}(-q)$ and $\chi_{P4}(q) \cong \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \bar{e}^*_3 - q \cdot \bar{f}^*_1 \wedge \bar{f}^*_2 \wedge \bar{f}^*_3 + \bar{f}^*_1 \wedge \bar{e}^*_2 \wedge \bar{f}^*_2 + \bar{f}^*_1 \wedge \bar{e}^*_3 \wedge \bar{f}^*_3$ (and so the latter trivector can be removed from the classification given in [8]).

(P4) $\chi_{P4}(q) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $q \in \mathbb{F}^*$;

(P5) $\chi_{P5}(q) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + \bar{f}_2^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + \bar{f}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $q \in \mathbb{F}^*$;

(P6) $\chi_{P6}(q,p) := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + p \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + p \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $p, q \in \mathbb{F}^*$;

(P7) $\chi_{P7} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;

(P8) $\chi_{P8} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(P9) $\chi_{P9} := \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(P10) $\chi_{P10} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$;

(P11) $\chi_{P11} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(P12) $\chi_{P12}(q) := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + q \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + q \cdot \bar{e}_3^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $q \in \mathbb{F}^*$;

(P13) $\chi_{P13} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(P14) $\chi_{P14}(q) := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + q \cdot \bar{f}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + q \cdot \bar{f}_3^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $q \in \mathbb{F}^*$;

(P15) $\chi_{P15} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$;

(P16) $\chi_{P16}(q) := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{f}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + q \cdot \bar{f}_3^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $q \in \mathbb{F}^*$;

(P17) $\chi_{P17}(q) := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + q \cdot \bar{e}_3^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $q \in \mathbb{F}^*$;

(P18) $\chi_{P18} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$;

(P19) $\chi_{P19} := \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_1^* \wedge \bar{f}_1^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$.

Theorem 5.2 ([8]) *Suppose* $\mathbb{F}$ *is an algebraically closed field of characteristic distinct from* 2.

- *Let* $i, j \in \{1, 2, \dots, 19\}$ *with* $i \neq j$. *Then no trivector of Type (Pi) is* $Sp(V, f)$*-equivalent with a trivector of Type (Pj).*
- *Let* $q_1, q_2 \in \mathbb{F}^*$ *and* $X \in \{P3, P5, P12, P14, P16, P17\}$. *Then the trivectors* $\chi_X(q_1)$ *and* $\chi_X(q_2)$ *of* V *are* $Sp(V, f)$*-equivalent if and only if* $q_2 \in \{q_1, -q_1\}$.
- *Let* $q_1, q_2 \in \mathbb{F}^*$. *Then the trivectors* $\chi_{P4}(q_1)$ *and* $\chi_{P4}(q_2)$ *of* V *are* $Sp(V, f)$*-equivalent if and only if* $q_2 = q_1$.
- *Let* $q_1, q_2, p_1, p_2 \in \mathbb{F}^*$. *Then the trivectors* $\chi_{P6}(q_1, p_1)$ *and* $\chi_{P6}(q_2, p_2)$ *of* V *are* $Sp(V, f)$*-equivalent if and only if* $(q_2, p_2) \in \{(q_1, p_1), (-q_1, -p_1)\}$.

6 Identification with the trivectors from Popov's classification

In the case $\mathbb{F}$ is a quadratically closed field of characteristic distinct from 2, we will establish explicit $Sp(V,f)$-equivalences between the trivectors mentioned in Corollary 4.1 and the nonzero trivectors mentioned in Theorem 5.1.

To achieve this goal, we will proceed as follows. Let χ be one of the trivectors mentioned in Corollary 4.1. Then there exists a unique trivector $\chi_1 \in W$ and a unique vector $\bar{v}_2 \in V$ such that $\chi = \chi_1 + \phi(\bar{v}_2)$. By Theorem 4.4, the trivector χ_1 is $Sp(V,f)$-equivalent with either 0, $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$, $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$, $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + q \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ (for some $q \in \mathbb{F}^*$) or $\bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$. We will determine an element $\theta \in Sp(V,f)$ which maps χ_1 to one of these five trivectors. We then have $\bigwedge^3(\theta)(\chi) = \bigwedge^3(\theta)(\chi_1) + \phi(\bar{v}_2^\theta)$. After calculating $\bar{v}_2^\theta$, we will be able to determine which trivector of Theorem 5.1 χ is $Sp(V,f)$-equivalent with.

The case of trivectors of Type (A1)

Let χ be the trivector $\chi_{A1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$, $\bar{v}_2 = \bar{o}$ and χ is $Sp(V,f)$-equivalent with χ_{P7}.

The case of trivectors of Type (A2)

Let χ be the trivector $\chi_{A2} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$. Then $\chi_1 = \frac{1}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_1^*$. Since $\chi_1 = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_2 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\bar{v}_2 = \bar{e}_3$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*, \frac{1}{2}\bar{e}_1^*, 2\bar{f}_1^*)$ of (V,f), we see that χ and $\chi_{P17}(1)$ are $Sp(V,f)$-equivalent.

The case of trivectors of Type (B1)

Let χ be the trivector $\chi_{B1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{f}_3^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \frac{1}{2}\bar{f}_3^* \wedge (\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_2^* \wedge \bar{f}_2^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{f}_3^*$. Let θ be the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(-2\bar{e}_1^*, -\frac{1}{2}\bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, -\frac{1}{2}\bar{f}_3^*, 2\bar{e}_3^*)$ of (V,f). Then $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2^\theta = \bar{e}_3^*$. So, χ_{B1} is $Sp(V,f)$-equivalent with $\chi_{P12}(1)$.

The case of trivectors of Type (B2)

Let χ be the trivector $\chi_{B2} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^*$. Then $\chi_1 = \frac{1}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) + \frac{1}{2}\bar{e}_3^* \wedge (\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_2^* \wedge \bar{f}_2^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_1^* + \frac{1}{2}\bar{e}_3^*$. Let θ be the element of $Sp(V,f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V,f) to the hyperbolic basis $(\bar{e}_1^* - \bar{e}_3^*, \frac{1}{2}(\bar{f}_1^* - \bar{f}_3^*), \bar{e}_2^*, \bar{f}_2^*, \bar{e}_1^* + \bar{e}_3^*, \frac{1}{2}(\bar{f}_1^* + \bar{f}_3^*))$. Then $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*$ and $\bar{v}_2^\theta = \bar{e}_1^*$. So, χ_{B_2} is $Sp(V,f)$-equivalent with χ_{P18}.

The case of trivectors of Type (B3)

Let χ be the trivector $\chi_{B3} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$. Then $\chi_1 = 0$, $\bar{v}_2 = \bar{e}_1^*$ and χ is $Sp(V,f)$-equivalent with χ_{P2}.

The case of trivectors of Type (B4)

Let χ be the trivector $\chi_{B4}(1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2 = 0$. By Proposition 4.5, $\chi_{B4}(1)$ and χ_{P15} are $Sp(V,f)$-equivalent.

The case of trivectors of Type (B5)

Let χ be the trivector $\chi_{B5}(\lambda) = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \frac{\lambda}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) + \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*)$ and $\bar{v}_2 = \frac{\lambda}{2}\bar{e}_1^*$. Now,

$$\chi_1 = \bar{e}_1^* \wedge \Big(\Big((\frac{\lambda}{2}+1)\bar{e}_2^* - \bar{e}_3^*\Big) \wedge \bar{f}_2^* + \Big(\bar{e}_2^* - (\frac{\lambda}{2}+1)\bar{e}_3^*\Big) \wedge \bar{f}_3^*\Big).$$

Suppose first that $\lambda = -4$. Then $\chi_1 = \bar{e}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_3^* - \bar{f}_2^*) = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\bar{v}_2 = \bar{e}_1$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\frac{\lambda}{2}\bar{e}_1^*, \frac{2}{\lambda}\bar{f}_1^*, \frac{2}{\lambda}(\bar{e}_2^* + \bar{e}_3^*), \frac{\lambda}{4}(\bar{f}_2^* + \bar{f}_3^*), \bar{f}_3^* - \bar{f}_2^*, \frac{1}{2}(\bar{e}_2^* - \bar{e}_3^*))$ of (V,f). So, in this case χ is $Sp(V,f)$-equivalent with χ_{P8}.

Suppose now that $\lambda \neq -4$. Let $\mu \in \mathbb{F}$ such that $\mu^2 = \frac{4}{\lambda(\lambda+4)}$. Put $\alpha := 1$, $\beta := -(\frac{\lambda}{2}+1) + \frac{1}{\mu}$, $\gamma := -\frac{1}{2}\mu$ and $\delta := \frac{1}{4}(2 + \mu(\lambda+2))$. Then $\alpha\delta - \beta\gamma = 1$, $\alpha\gamma = -\frac{1}{2}\mu$, $\beta\delta = -\frac{1}{2}\mu$, $\alpha\delta = \frac{1}{2}(1 + \mu(\frac{\lambda}{2}+1))$ and $\beta\gamma = \frac{1}{2}(\mu(\frac{\lambda}{2}+1) - 1)$. We have $\chi_1 = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_2 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\bar{v}_2 = \frac{\lambda\mu}{2}\bar{e}_3$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\alpha\bar{e}_2^* + \beta\bar{e}_3^*, \delta\bar{f}_2^* - \gamma\bar{f}_3^*, \gamma\bar{e}_2^* + \delta\bar{e}_3^*, -\beta\bar{f}_2^* + \alpha\bar{f}_3^*, \frac{1}{\mu}\bar{e}_1^*, \mu\bar{f}_1^*)$ of (V,f). So, χ is $Sp(V,f)$-equivalent with $\chi_{P_{17}}(\frac{\lambda\mu}{2})$.

Observe that $(\frac{\lambda\mu}{2})^2 = \frac{\lambda}{\lambda+4}$ can take any value in $\mathbb{F}^*$ except for 1. So, $\frac{\lambda\mu}{2}$ can take any value in $\mathbb{F}^*$, except for 1 and -1. We have already seen above that the trivector $\chi_{P17}(1)$ (and hence also $\chi_{P17}(-1)$) is $Sp(V,f)$-equivalent with χ_{A2}.

The case of trivectors of Type (C1)

Let χ be the trivector $\chi_{C1}(\lambda) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2 = \bar{o}$. The trivector χ is $Sp(V,f)$-equivalent with $\chi_{P3}(\lambda)$.

The case of trivectors of Type (C2)

Let χ be the trivector $\chi_{C2}(\lambda) = \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \frac{\lambda}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*)$ and $\bar{v}_2 = \frac{\lambda}{2}\bar{e}_1^*$. We have

$$\chi_1 = (\bar{f}_1^* + \frac{\lambda}{4}\bar{e}_1^*) \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \frac{\lambda}{4} \cdot \bar{e}_1^* \wedge (\bar{e}_2^* - \bar{e}_3^*) \wedge (\bar{f}_2^* + \bar{f}_3^*).$$

From this, we deduce that $\chi_1 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \lambda \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$ and $\bar{v}_2 = \bar{e}_1$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\frac{\lambda}{2}\bar{e}_1^*, \frac{2}{\lambda}\bar{f}_1^* + \frac{1}{2}\bar{e}_1^*, \frac{1}{2}(\bar{e}_2^* - \bar{e}_3^*), \bar{f}_2^* - \bar{f}_3^*, \bar{f}_2^* + \bar{f}_3^*, -\frac{1}{2}(\bar{e}_2^* + \bar{e}_3^*))$ of (V, f). So, χ is $Sp(V, f)$-equivalent with $\chi_{P4}(\lambda)$.

The case of trivectors of Type (C3)

Let χ be the trivector $\chi_{C3}(\lambda) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \frac{1}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) + \frac{\lambda}{2}\bar{f}_1^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_2^* \wedge \bar{f}_2^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_1^* + \frac{\lambda}{2}\bar{f}_1^*$. We have $\chi_1 = (\frac{1}{2}\bar{e}_1^* - \frac{\lambda}{2}\bar{f}_1^*) \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{f}_2 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\bar{v}_2 = \frac{\lambda}{2}\bar{f}_3$ where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*, \frac{1}{2}\bar{e}_1^* - \frac{\lambda}{2}\bar{f}_1^*, \frac{1}{\lambda}\bar{e}_1^* + \bar{f}_1^*)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with $\chi_{P16}(\frac{\lambda}{2})$.

The case of trivectors of Type (C4)

Let χ be the trivector $\chi_{C4}(\lambda) = \bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{e}_2^* + \frac{1}{2}\bar{f}_1^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_2^* \wedge \bar{f}_2^*) + \frac{\lambda}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{f}_1^* + \frac{\lambda}{2}\bar{e}_1^*$. We have

$$\chi_1 = \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{e}_2^* + \bar{e}_2^* \wedge \bar{f}_2^* \wedge (\frac{\lambda}{2}\bar{e}_1^* - \frac{1}{2}\bar{f}_1^*) + \bar{e}_3^* \wedge \bar{f}_3^* \wedge (\frac{1}{2}\bar{f}_1^* - \frac{\lambda}{2}\bar{e}_1^*).$$

We calculate that $\chi_1 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_2$ and $\bar{v}_2 = \bar{e}_3 + \frac{\lambda}{2}\bar{f}_3$ where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(-\frac{\lambda}{2}\bar{e}_2^*, -\frac{2}{\lambda}\bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*, \frac{\lambda}{2}\bar{e}_1^* - \frac{1}{2}\bar{f}_1^*, \frac{2}{\lambda}\bar{f}_1^*)$ of (V, f). We have $\chi_1 = \bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \bar{e}_1' \wedge \bar{f}_1' \wedge \bar{e}_3' + \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{f}_2'$ and $\bar{v}_2 = \frac{\lambda}{2}\bar{f}_3'$, where $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is the hyperbolic basis $(\bar{e}_1, \bar{f}_1 - \frac{1}{\lambda}\bar{e}_2, \bar{e}_2, \bar{f}_2 - \frac{1}{\lambda}\bar{e}_1, \bar{e}_3, \bar{f}_3 + \frac{2}{\lambda}\bar{e}_3)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with $\chi_{P14}(\frac{\lambda}{2})$.

The case of trivectors of Type (C5)

Let χ be the trivector $\chi_{C5}(\lambda) = \bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge \bar{f}_1^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_2^* + \frac{1}{2}\bar{e}_1^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_2^* \wedge \bar{f}_2^*) - \frac{\lambda}{2}\bar{e}_2^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_1^* \wedge \bar{f}_1^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_1^* - \frac{\lambda}{2}\bar{e}_2^*$. Let θ be the element of $Sp(V, f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f) to the hyperbolic basis $(2\bar{e}_1^*, -\frac{1}{\lambda}\bar{e}_1^* + \frac{1}{2\lambda}\bar{e}_3^* + \frac{1}{2}\bar{f}_1^* + \frac{1}{\lambda}\bar{f}_2^* + \frac{1}{2}\bar{f}_3^*, -\frac{2}{\lambda}\bar{f}_2^*, -\frac{1}{2}\bar{e}_1^* + \frac{\lambda}{2}\bar{e}_2^* - \frac{1}{2}\bar{f}_2^* + \frac{\lambda}{2}\bar{f}_3^*, \frac{1}{\lambda}\bar{e}_1^* - \frac{1}{\lambda}\bar{e}_3^* - \frac{1}{\lambda}\bar{f}_2^*, -\bar{e}_1^* - \lambda\bar{f}_3^*)$ of (V, f). Then $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2^\theta = \bar{e}_1^* + \bar{f}_2^*$. So, χ is $Sp(V, f)$-equivalent with $\chi_{P5}(\lambda)$.

The case of trivectors of Type (C6)

Let $\lambda \in \mathbb{F}^*$, $\epsilon \in \mathbb{F} \setminus \{0, -1\}$ and let χ be the trivector $\chi_{C6}(\lambda, \epsilon) = \bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon\bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$. Then $\chi_1 = \epsilon \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_2^* + \frac{1}{2}\bar{f}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) + \frac{\epsilon}{2}\bar{f}_1^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_2^* \wedge \bar{f}_2^*) + \frac{\lambda}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{f}_1^* + \frac{\epsilon}{2}\bar{f}_1^* + \frac{\lambda}{2}\bar{e}_1^*$. Now

$$\begin{aligned} \chi_1 &= -\frac{1}{4\eta} \cdot (\lambda\bar{e}_1^* + (\eta - 1)^2\bar{f}_1^*) \wedge (\bar{e}_3^* - \eta\bar{e}_2^*) \wedge (\eta\bar{f}_3^* + \bar{f}_2^*) \\ &\quad + \frac{1}{4\eta} \cdot (\lambda\bar{e}_1^* + (\eta + 1)^2\bar{f}_1^*) \wedge (\bar{e}_3^* + \eta\bar{e}_2^*) \wedge (-\eta\bar{f}_3^* + \bar{f}_2^*) \end{aligned}$$

where $\eta \in \mathbb{F}^*$ such that $\eta^2 = -\epsilon$. One calculates that $\chi_1 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \eta\lambda \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$ and $\bar{v}_2 = \bar{e}_1 + \frac{\epsilon+1}{4}(\eta\lambda) \cdot \bar{f}_1$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis of (V, f) defined by

$$\bar{e}_1 := \frac{\eta+1}{4}\Big(\lambda\bar{e}_1^* + (\eta-1)^2\bar{f}_1^*\Big), \quad \bar{e}_2 := \frac{\eta\bar{e}_2^* - \bar{e}_3^*}{\eta}, \quad \bar{e}_3 := \frac{\eta\bar{f}_3^* + \bar{f}_2^*}{\eta+1},$$

$$\bar{f}_1 := \frac{\lambda\bar{e}_1^* + (\eta+1)^2\bar{f}_1^*}{(\eta+1)\eta\lambda}, \quad \bar{f}_2 := \frac{-\eta\bar{f}_3^* + \bar{f}_2^*}{2}, \quad \bar{f}_3 := -\frac{\eta+1}{2\eta}(\bar{e}_3^* + \eta\bar{e}_2^*).$$

So, χ is $Sp(V, f)$-equivalent with the trivector $\chi_{P6}(\eta\lambda, \eta\lambda\frac{\epsilon+1}{4})$.

Observe that for all $q, p \in \mathbb{F}^*$ with $p \neq \frac{q}{4}$, it is possible to choose $\epsilon \in \mathbb{F}^* \setminus \{0, -1\}$ and $\lambda, \eta \in \mathbb{F}^*$ such that $\eta^2 = -\epsilon$, $q = \eta\lambda$ and $p = \eta\lambda \cdot \frac{\epsilon+1}{4}$. We will later see that for every $q \in \mathbb{F}^*$, the trivector $\chi_{P6}(q, \frac{q}{4})$ is $Sp(V, f)$-equivalent with $\chi_{D2}(-q^2)$.

The case of trivectors of Type (D1)

Let χ be the trivector $\chi_{D1} = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$. Then $\chi_1 = \frac{1}{2}\bar{e}_1^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_3^* \wedge \bar{f}_3^*) + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* - \frac{1}{2}\bar{f}_3^* \wedge (\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_2^* \wedge \bar{f}_2^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_1^* - \frac{1}{2}\bar{f}_3^*$. Let θ be the element of $Sp(V, f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f) to the hyperbolic basis $(\bar{f}_1^* - \bar{e}_3^*, -\frac{1}{2}(\bar{e}_1^* + \bar{f}_3^*), 2\bar{e}_2^*, \frac{1}{2}\bar{f}_2^*, \frac{1}{2}(\bar{f}_3^* - \bar{e}_1^*), -\bar{f}_1^* - \bar{e}_3^*)$ of (V, f). Then $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2^\theta = \bar{f}_1^*$. So, χ is $Sp(V, f)$-equivalent with χ_{P13}.

The case of trivectors of Type (D2)

Let χ be the trivector $\chi_{D2}(\lambda) = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* - \frac{1}{2}\bar{f}_2^* \wedge (\bar{e}_1^* \wedge \bar{f}_1^* - \bar{e}_3^* \wedge \bar{f}_3^*)$ and $\bar{v}_2 = -\frac{1}{2}\bar{f}_2^*$. If $\eta \in \mathbb{F}$ such that $\lambda = -\eta^2$, then

$$\chi_1 = \frac{1}{2} \cdot (\bar{e}_2^* - \frac{1}{2\eta}\bar{f}_2^*) \wedge (\bar{f}_1^* + \eta\bar{f}_3^*) \wedge (\bar{e}_3^* - \eta\bar{e}_1^*) + \frac{1}{2} \cdot (\bar{e}_2^* + \frac{1}{2\eta}\bar{f}_2^*) \wedge (\bar{f}_1^* - \eta\bar{f}_3^*) \wedge (\bar{e}_3^* + \eta\bar{e}_1^*).$$

We have that $\chi_1 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 - \eta \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$ and $\bar{v}_2 = \bar{e}_1 - \frac{\eta}{4}\bar{f}_1$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\frac{\eta}{2}\bar{e}_2^* - \frac{1}{4}\bar{f}_2^*, 2\bar{e}_2^* + \frac{1}{\eta}\bar{f}_2^*, \frac{1}{\eta}\bar{f}_1^* + \bar{f}_3^*, -\frac{1}{2}(\bar{e}_3^* + \eta\bar{e}_1^*), \bar{e}_3^* - \eta\bar{e}_1^*, \frac{1}{2\eta}(\eta\bar{f}_3^* - \bar{f}_1^*))$ of (V, f). So, χ is $Sp(V, f)$-equivalent with $\chi_{P6}(-\eta, -\frac{\eta}{4})$.

The case of trivectors of Type (D3)

Let χ be the trivector $\chi_{D3}(1, 1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ and $\bar{v}_2 = 0$. By Proposition 4.6, χ is $Sp(V, f)$-equivalent with χ_{P10}.

The case of trivectors of Type (D4)

Let χ be the trivector $\chi_{D4}(\lambda, 1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ for some $\lambda \in \mathbb{F}^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \frac{\lambda}{2}\bar{e}_2^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_1^* \wedge \bar{f}_1^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ and $\bar{v}_2 = \frac{\lambda}{2}\bar{e}_2^*$. We have $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge (\frac{\lambda}{2}\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\lambda\bar{f}_1^* + \frac{\lambda}{2}\bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$.

Suppose first that $\lambda = 4$. Then $\chi_1 = (\bar{e}_1^* - 2\bar{e}_3^*) \wedge (-\bar{e}_3^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge (2\bar{f}_1^* + \bar{f}_3^*))$. Let $\mu \in \mathbb{F}$ such that $\mu^2 = -1$ and let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ be the following hyperbolic basis of (V, f):

$$(\bar{e}_2^* - \mu\bar{e}_3^*, \frac{1}{2}\bar{f}_2^* + \frac{\mu}{2}(2\bar{f}_1^* + \bar{f}_3^*), \frac{\mu}{2}\bar{e}_2^* - \frac{1}{2}\bar{e}_3^*, -\mu\bar{f}_2^* - (2\bar{f}_1^* + \bar{f}_3^*), \frac{1}{\mu}(\bar{e}_1^* - 2\bar{e}_3^*), \mu\bar{f}_1^*).$$

Then $\chi_1 = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_2$ and $\bar{v}_2 = \frac{\lambda}{4\mu}(\mu\bar{e}_1 + 2\bar{e}_2)$. Putting $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3') = (\frac{\lambda}{4}\bar{e}_1, \frac{4}{\lambda}\bar{f}_1, \frac{\lambda}{2\mu}\bar{e}_2, \frac{2\mu}{\lambda}\bar{f}_2, \bar{e}_3, \bar{f}_3)$, we see that $\chi_1 = \bar{e}_1' \wedge \bar{f}_1' \wedge \bar{e}_3' + \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{f}_2'$ and $\bar{v}_2 = \bar{e}_1' + \bar{e}_2'$. So, χ is $Sp(V, f)$-equivalent with χ_{P19}.

Next, we suppose that $\lambda \neq 4$. Let $\mu \in \mathbb{F}$ such that $\mu^2 = \frac{\lambda(\lambda-4)}{4}$ and put $\alpha := \frac{\lambda}{2} + \mu$, $\beta := \frac{\lambda}{4\mu} - \frac{1}{2}$, $\gamma := 1$ and $\delta := \frac{1}{2\mu}$. Then $\alpha\delta - \beta\gamma = 1$. If θ denotes the element of $Sp(V, f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f) to the hyperbolic basis $(\alpha\bar{e}_1^* + \beta\bar{e}_3^*, \delta\bar{f}_1^* - \gamma\bar{f}_3^*, \bar{e}_2^*, \bar{f}_2^*, \gamma\bar{e}_1^* + \delta\bar{e}_3^*, -\beta\bar{f}_1^* + \alpha\bar{f}_3^*)$ of (V, f), then $\bigwedge^3(\theta)(\chi_1) = \mu \cdot \bar{e}_2^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_1^* \wedge \bar{f}_1^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$ and $\bar{v}_2^\theta = \frac{\lambda}{2}\bar{e}_2^*$. So, $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_3 + \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_2 + \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3$ and $\bar{v}_2^\theta = \frac{\lambda}{2\mu}\bar{e}_3$, where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\bar{e}_3^*, \bar{f}_3^*, \mu\bar{e}_1^*, \frac{1}{\mu}\bar{f}_1^*, \mu\bar{e}_2^*, \frac{1}{\mu}\bar{f}_2^*)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with $\chi_{P12}(\frac{\lambda}{2\mu})$.

Observe that $(\frac{\lambda}{2\mu})^2 = \frac{\lambda}{\lambda-4}$ can take any value in $\mathbb{F}^*$ except for 1. So, $\frac{\lambda}{2\mu}$ can take any value in $\mathbb{F}^*$, except for 1 and -1. We have already seen above that the trivector $\chi_{P12}(1)$ (and hence also $\chi_{P12}(-1)$) is $Sp(V, f)$-equivalent with χ_{B1}.

The case of trivectors of Type (D5)

Let χ be the trivector $\chi_{D5}(1) = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*$. Then $\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^* - \frac{1}{2}\bar{e}_3^* \wedge (\bar{e}_2^* \wedge \bar{f}_2^* - \bar{e}_1^* \wedge \bar{f}_1^*) + \frac{1}{2}\bar{e}_2^* \wedge (\bar{e}_3^* \wedge \bar{f}_3^* - \bar{e}_1^* \wedge \bar{f}_1^*)$ and $\bar{v}_2 = \frac{1}{2}\bar{e}_2^* - \frac{1}{2}\bar{e}_3^*$. We have

$$\chi_1 = \bar{e}_1^* \wedge \bar{e}_2^* \wedge (\frac{1}{2}\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \frac{1}{2}\bar{f}_2^* + \frac{1}{2}\bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge (\frac{1}{2}\bar{f}_1^* - \bar{f}_2^*).$$

Let θ be the element of $Sp(V, f)$ mapping the hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f) to the hyperbolic basis $(-\bar{e}_3^* - \bar{e}_1^*, -\bar{f}_3^*, \bar{e}_1^* - \frac{1}{2}\bar{e}_2^*, \frac{1}{2}\bar{f}_1^* - \bar{f}_2^* - \frac{1}{2}\bar{f}_3^*, -\bar{e}_1^* - \frac{1}{2}\bar{e}_2^*, -\frac{1}{2}\bar{f}_1^* - \bar{f}_2^* + \frac{1}{2}\bar{f}_3^*)$ of (V, f). Then $\bigwedge^3(\theta)(\chi_1) = \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_2^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*$ and $\bar{v}_2^\theta = \bar{e}_1^*$. So, χ is $Sp(V, f)$-equivalent with χ_{P11}.

The case of trivectors of Type (D6)

Let χ be the trivector $\chi_{D6} = -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*$. Then $\chi_1 = \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^*$ and $\bar{v}_2 = -\bar{e}_1^*$. We have that $\chi_1 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3$ and $\bar{v}_2 = \bar{f}_1$ where $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ is the hyperbolic basis $(\bar{f}_1^*, -\bar{e}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f). So, χ is $Sp(V, f)$-equivalent with χ_{P9}.

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